Notes on CDP (2019)

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The Household Decision Problem and Migration

- Simplify and assume one sector $J = 1$, and N regions.
- Mobility costs are $\tau^{n,i} \geq 0$ with $\tau^{n,n} = 0$.
- Each household provides one unit of labor.
- Consumption is equal to real income:

$$C_t^n = \frac{w_t^n}{P_t^n}$$

The Household Decision Problem and Migration

- The value function of a worker in location n at time t is given by:

$$\begin{aligned} v_t^n &= \ln C_t^n + \max_i \{\beta E[v_{t+1}^i] \; - \; \tau^{n,i} + \nu \epsilon_t^i\} \\ &= \max_i \ln \frac{w_t^n}{P_t^n} \; - \; \tau^{n,i} \; + \; \nu \epsilon_t^i \; + \; \beta E[v_{t+1}^i] \end{aligned}$$

- The idiosyncratic errors are Type I extreme value, and hence the expected value function is given by:

$$E[v_t^n] \; = \; V_t^n \; = \ln \frac{w_t^n}{P_t^n} \; + \; \nu \; \ln \left(\sum_{i=1}^{N} \exp(\beta V_{t+1}^i \; - \; \tau^{n,i})^{1/\nu} \right)$$

The Household Decision Problem and Migration

- Hence, the fraction of households that relocate from n to i is given by the dynamic logit expression:

$$\mu_t^{n,i} = \frac{\exp(\beta V_{t+1}^i - \tau^{n,i})^{1/\nu}}{\sum_{j=1}^N \exp(\beta V_{t+1}^j - \tau^{n,j})^{1/\nu}}$$

- The law of motion for the population is given by

$$L_{t+1}^i = \sum_{n=1}^N \mu_t^{n,i} L_t^n$$

Dynamic Hat Algebra

- Define $\hat{y}_{t+1} = \frac{y_{t+1}}{y_t}$.
- Define $u_t^n = \exp(V_t^n)$.
- Consider an exogenous sequence of wages and price levels $\{w_s^n,\ P_s^n\}_{n=1}^N$ for $s = t, t+1, t+2, ..., \bar{t}$.
- Note that this sequence can be generated from fundamentals via an EK type trade model as discussed below.

Dynamic Hat Algebra

- The evolution of the populations over time is the solution to the following system of equations:

$$\mu_{t+1}^{n,i} = \frac{\mu_t^{n,i} \left(\hat{u}_{t+2}^{i}\right)^{\beta/\nu}}{\sum_{j=1}^{N} \mu_t^{n,j} \left(\hat{u}_{t+2}^{j}\right)^{\beta/\nu}} \quad (1)$$

$$\hat{u}_{t+1}^{n} = \frac{\hat{w}_{t+1}^{n}}{\hat{P}_{t+1}^{n}} \left(\sum_{i=1}^{N} \mu_t^{n,i} \left(\hat{u}_{t+2}^{i}\right)^{\beta/\nu} \right)^{\nu} \quad (2)$$

$$L_{t+1}^{i} = \sum_{n=1}^{N} \mu_t^{n,i} L_t^{n} \quad (3)$$

- These equations are derived in in the posted note.

Closing The Model

- To close the model would be to specify a static trade model that gives us a system of equations to solve for wages w_t^n and price levels P_t^n as a function of fundamentals such as iceberg costs and location specific productivities.
- We can solve a sequence of the equilibria using static hat algebra as long as we know the law of motion for the populations.
- Consider the EK model without intermediate goods as discussed in the MIT lecture notes. We only need to add time subscripts t to the notation.
- Recall, that each region i at time t has a distribution of productivities given by:

$$F_t^i(z) = \exp(-T_t^i z^{-\theta})$$

Closing the Model

- $d_t^{ni} \geq 1$ units need to be shipped from i so that 1 unit makes it to n (iceberg costs).
- In the model without intermediate goods $c_t^i = w_t^i / P_t^i$ because of constant returns to scale.
- Recall that the price distribution faced by region n is

$$G_t^n(p) = 1 - \exp(-\Phi_t^n p^\theta)$$

where

$$\Phi_t^n = \sum_{i=1}^{N} T_t^i \left(w_t^i d_t^{ni}\right)^{-\theta}$$

which is also the price distribution of prices that region n buys from any region i, i.e. all exporters face the same price distribution in equilibrium.

Closing the Model

- Region n buys good from region i if $i = \arg\min\{p_1, ..., p_N\}$ and the shares are given by

$$\pi_t^{ni} = \frac{T_t^i (w_t^i d_t^{ni})^{-\theta}}{\Phi_t^n}$$

- The overall price index in region n is

$$\begin{aligned} P_t^n &= \left(\int_0^1 p(u)^{1-\sigma} \, dG_t^n(u) \right)^{\frac{1}{1-\sigma}} \\ &= \gamma \left(\Phi_t^n \right)^{-\frac{1}{\theta}} \\ &= \gamma \left(\sum_{i=1}^{N} T_t^i \left(w_t^i d_t^{ni} \right)^{-\theta} \right)^{-\frac{1}{\theta}} \end{aligned}$$

where

$$\gamma = \left[\Gamma \left(\frac{1-\sigma}{\theta} + 1 \right) \right]^{1/(1-\sigma)}$$

Closing the Model

- In this simple model without intermediate goods the price index in region n is a function of the exogenous fundamentals as well as the endogenous wages in all regions.
- Let X_t^{ni} be total spending of region n on goods from region i.
- Total spending of region n is $X_t^n = \sum_{i=1}^N X_t^{ni}$.
- Then

$$\frac{X_t^{ni}}{X_t^n} = \pi_t^{ni} = \frac{T_t^i (w_t^i d_t^{ni})^{-\theta}}{\Phi_t^n}$$

or

$$X_t^{ni} = \frac{T_t^i (w_t^i d_t^{ni})^{-\theta}}{\Phi_t^n} X_t^n$$

Closing the Model

- In equilibrium, income of region n must equal total expenditures on goods from region n (zero trade deficit):

$$Y_t^n = w_t^n L_t^n = \sum_{i=1}^{N} X_t^{in}$$

and hence we have:

$$\begin{aligned} w_t^n L_t^n &= \sum_{i=1}^{N} X_t^{in} \\ &= \sum_{i=1}^{N} \frac{T_t^n (w_t^n d_t^{in})^{-\theta}}{\sum_{j=1}^{N} T_t^j \, (w_t^j d_t^{ij})^{-\theta}} \, w_t^i L_t^i \end{aligned}$$

These are $N-1$ equations in N unknowns. Hence we can compute the equilibrium wages from that system of equations up to a normalization if we know $L_t^1, \ldots L_t^N$.

Hat Alegra in Eaton-Kortum

Finally, recall from the previous lecture on the EK model that we can solve for a sequence of static equilibria using static hat algebra.In particular, we have:

$$\hat{w}^n_{t+1}\hat{L}^n_{t+1}Y^n_t = \sum_{i=1}^{N} \frac{\pi^{in}_t \ \hat{T}^n_{t+1} \left(\hat{w}^n_{t+1}\hat{d}^{in}_{t+1}\right)^{-\theta}}{\sum_{j=1}^{N} \pi^{ij}_t \ \hat{T}^j_{t+1} \left(\hat{w}^j_{t+1}\hat{d}^{ij}_{t+1}\right)^{-\theta}} \ \hat{w}^i_{t+1}\hat{L}^i_{t+1} \ Y^i_t \quad (4)$$

and

$$\hat{P}^n_{t+1} = \left(\sum_{i=1}^{N} \pi^{ni}_t \ \hat{T}^i_{t+1} \left(\hat{w}^i_{t+1}\hat{d}^{ni}_t\right)^{-\theta}\right)^{-\frac{1}{\theta}} \quad (5)$$

and

$$\hat{\pi}^{ni}_{t+1} = \frac{\hat{T}^i_{t+1}\left(\hat{w}^i_{t+1}\hat{d}^{ni}_{t+1}\right)^{-\theta}}{\sum_{j=1}^{N} \pi^{nj}_t \hat{T}^j_{t+1} \left(\hat{w}^j_{t+1}\hat{d}^{nj}_{t+1}\right)^{-\theta}} \quad (6)$$

The Main Result in CDP

Given an initial allocation of labor $L_0^1, ...L_0^N$ and wages $w_0^1, ...w_0^N$ and an exogenous sequence of changes in fundamentals (iceberg costs $\hat{d}_1, \hat{d}_2, ...\hat{d}_{\bar{t}}$ and productivities $\hat{T}_1, \hat{T}_2, ... \hat{T}_{\bar{t}}$), assume that the economy reaches a steady state in $\bar{t}$ and $\hat{u}_t^i = 1$ for all $t \geq \bar{t}$. Then the evolution of the economy is completely characterized by a solution to the system of equations given by equations (1) - (6).

This result then implies that we can compute an equilibrium using a clever algorithm.

Take as given the set of initial conditions $(L_0, \mu_{-1}, w_0, P_0, \pi_0)$.
Note that given L_0 and w_0, we know Y_0.
Suppose you are at iteration step k, update all variables as follows:

1. For all $t \geq 0$, use $\left\{\hat{u}^n_{t+1}(k)\right\}_{t=0}^{\bar{t}}$ and $\mu^{n,i}_{-1}$ to solve for the path of $\left\{\mu^{n,i}_t(k+1)\right\}_{t=0}^{T}$ using equation (1).
2. Use the path for $\left\{\mu^{n,i}_t(k+1)\right\}_{t=0}^{T}$ and L^n_0 to get the path for $\left\{L^n_{t+1}(k+1)\right\}_{t=0}^{T}$ using equation (3).
3. Given the sequence $\left\{L^n_{t+1}(k+1)\right\}_{t=0}^{\bar{t}}$, Y_0, π_0, use equations (4) - (6) to compute the sequences of $\left\{\hat{w}^n_{t+1}(k+1)\right\}_{t=0}^{\bar{t}}$, $\left\{\hat{P}^n_{t+1}(k+1)\right\}_{t=0}^{\bar{t}}$ and $\left\{\hat{\pi}^n_{t+1}(k+1)\right\}_{t=0}^{\bar{t}}$.
4. For each t, use $\mu^{n,i}_t$(k+1), $\hat{w}^n_{t+1}(k+1)$, $\hat{P}^n_{t+1}(k+1)$, and $\hat{u}^n_{t+2}(k)$ to solve backwards for $\hat{u}^n_{t+1}(k+1)$ using equation (2). This delivers a new path for $\left\{\hat{u}^n_{t+1}(k+1)\right\}_{t=0}^{\bar{t}}$.
5. Check if $\left\{\hat{u}^n_{t+1}(k+1)\right\}_{t=0}^{\bar{t}} \simeq \left\{\hat{u}^n_{t+1}(k)\right\}_{t=0}^{\bar{t}}$. The path should converge to $\hat{u}^n_{t+1} = 1$ for a sufficiently large $\bar{t}$. If it is not, continue to iterate.