Discrete Choice: Frechet Errors

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The Frechet Distribution

- The Frechet distribution is given by

$$\begin{aligned} F(z) &= e^{-T\,z^{-\epsilon}} \\ f(z) &= T\,\epsilon\,z^{-\epsilon-1}\,e^{-T\,z^{-\epsilon}} \end{aligned}$$

  with $T > 0$ and $\epsilon > 1$.
- T is called the level and ϵ is called the dispersion of the distribution.

Properties of the Frechet Distribution

- Note that the mean of the distribution is given by

$$E(Z) = \Gamma\left(\frac{\epsilon - 1}{\epsilon}\right) T^{1/\epsilon}$$

  where the gamma function is given by $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$.

- For any constant k

$$k\, Z = F(Tk^{\epsilon}, \epsilon)$$

- Suppose that Z_1 and Z_2 are independently, Frechet distributed with common dispersion parameter ϵ but different levels T_1 and T_2. We can show that

$$\begin{aligned} Pr\{Z_1 < Z_2\} &= \int_0^\infty \int_0^{z_2} f_1(z_1)\, dz_1\, f_2(z_2)\, dz_2 \\ &= \int_0^\infty F_1(z_2)\, f_2(z_2)\, dz_2 \\ &= \frac{T_1}{T_1 + T_2} \end{aligned}$$

More Generally

Let $Z_1, ... Z_n$ be independent random variables where

$$Z_i \quad \sim \quad F(T_i, \epsilon)$$

Then the probability that

$$Pr\{Z_i \geq \max_{j \neq i} Z_j\} \quad = \quad \frac{T_i}{\sum_{j=1}^{n} T_j}$$

i.e. the Frechet distribution is max-stable:

$$max \ \{Z_1, ..., Z_n\} \quad \sim \quad F(\sum T_i, \epsilon)$$

An Example: Mono-centric City

Suppose utility of individual i in location j is given by

$$u_{ij} = (w - R_j - \tau d_j)\, z_{ij}$$

where

- w is the wage in the city,
- R_j is the rent,
- $d_j = j$ is the distance to the CBD,
- τ is commuting cost parameter,
- z_{ij} is iid $F(T, \epsilon)$.

An Example: Mono-centric City

Then

$$u_{ij} \quad \sim \quad F(T(w - R_j - \tau j)^{\epsilon}, \epsilon)$$

And hence the market share of each community is given by

$$\pi_j \quad = \quad \frac{(w - R_j - \tau j)^{\epsilon}}{\sum_{i=1}^{J} (w - R_i - \tau i)^{\epsilon}}$$

An Example: Mono-centric City

The expected utility of living in the city

$$E[\max\ u_{ij}] \;\sim\; F(T\sum_{j=1}^{J}(w - R_j - \tau j)^{\epsilon}, \epsilon)$$

and the expected utility of living in the city is:

$$E[\max u_{ij}] \;=\; \Gamma\left(\frac{\epsilon - 1}{\epsilon}\right)[T\ \sum_{j=1}^{J}(w - R_j - \tau j)^{\epsilon}]^{1/\epsilon}$$

The Generalized Gumbel Distribution

Recall that the Generalized Gumbel or Generalized Extreme Value Distribution is given by

$$F(z) = e^{-e^{-z/\mu}}$$

for some scale parameter μ.

Choice Probabilities with Additive Separable Utilities

Let $U_i = V_i + Z_i$ for $i = 1, ...,$ some deterministic V_i and Z_i i.i.d. GEV. Then the choice probability satisfies:

$$Pr\{U_i \geq \max_{j \neq i} U_j\} \quad = \quad \frac{exp(V_i/\mu)}{\sum_{j=1}^{n} \; exp(V_j/\mu)}$$

Define $W_i = \exp(V_i)$, then we have

$$Pr\{U_i \geq \max_{j \neq i} U_j\} \quad = \quad \frac{W_i^{\frac{1}{\mu}}}{\sum_{j=1}^{n} \; W_j^{\frac{1}{\mu}}}$$

The choice probabilities are thus similar to those we get with Frechet shocks.