Notes on CDP (2019)

Holger Sieg

University of Pennsylvania

April 4, 2024

1 The Household Decision Problem and Migration

- Simplify and assume one sector $J = 1$ (which will be omitted for the purposes of the notes), and N regions.

- Mobility costs are $\tau^{n,i} \geq 0$ with $\tau^{n,n} = 0$.

- Each household provides one unit of labor. Consumption is equal to real income:

$$C_t^n = \frac{w_t^n}{P_t^n}$$

- The value function of a worker in location n at time t is given by:

$$\begin{aligned} v_t^n &= \ln C_t^n + \max_i \{\beta E[v_{t+1}^i] - \tau^{n,i} + \nu \epsilon_t^i\} \\ &= \max_i \ln \frac{w_t^n}{P_t^n} - \tau^{n,i} + \nu \epsilon_t^i + \beta E[v_{t+1}^i] \end{aligned}$$

where the last one is the more common notation in Rust.

- The idiosyncratic errors are Type I extreme value, and hence the expected value function is given by:

$$E[v_t^n] = V_t^n = \ln \frac{w_t^n}{P_t^n} + \nu \ln \left(\sum_{i=1}^N \exp(\beta V_{t+1}^i - \tau^{n,i})^{1/\nu} \right)$$

- Hence, the fraction of households that relocate from n to i is given by the dynamic logit expression:

$$\mu_t^{n,i} = \frac{\exp(\beta V_{t+1}^i - \tau^{n,i})^{1/\nu}}{\sum_{j=1}^N \exp(\beta V_{t+1}^j - \tau^{n,j})^{1/\nu}}$$

- The law of motion for the population is given by

$$L_{t+1}^i = \sum_{n=1}^N \mu_t^{n,i} L_t^n$$

2 Dynamic Hat Algebra

- Define $\hat{y}_{t+1} = \frac{y_{t+1}}{y_t}$.
- Define $u_t^n = \exp(V_t^n)$.
- Consider an exogenous sequence of wages and price levels $\{w_s^n,\ P_s^n\}_{n=1}^N$ for $s = t, t+1, t+2, ..., \bar{t}$..
- Note that this sequence can be generated from fundamentals via an EK type trade model as discussed below.
- The evolution of the populations over time is the solution to the following system of equations:

$$\begin{aligned}
\mu_{t+1}^{n,i} &= \frac{\mu_t^{n,i}\ (\hat{u}_{t+2}^i)^{\beta/\nu}}{\sum_{j=1}^N\ \mu_t^{n,j}\ (\hat{u}_{t+2}^j)^{\beta/\nu}} && (1)\\
\hat{u}_{t+1}^n &= \frac{\hat{w}_{t+1}^n}{\hat{P}_{t+1}^n}\left(\sum_{i=1}^N\ \mu_t^{n,i}\ (\hat{u}_{t+2}^i)^{\beta/\nu}\right)^{\nu} && (2)\\
L_{t+1}^i &= \sum_{n=1}^N\ \mu_t^{n,i}\ L_t^n && (3)
\end{aligned}$$

- To derive the equation (1), note that

$$\begin{aligned}
\frac{\mu_{t+1}^{n,i}}{\mu_t^{n,i}} &= \frac{\frac{\exp(\beta V_{t+2}^i\ -\ \tau^{n,i})^{1/\nu}}{\sum_{j=1}^N\ \exp(\beta V_{t+2}^j\ -\ \tau^{n,j})^{1/\nu}}}{\frac{\exp(\beta V_{t+1}^i\ -\ \tau^{n,i})^{1/\nu}}{\sum_{j=1}^N\ \exp(\beta V_{t+1}^j\ -\ \tau^{n,j})^{1/\nu}}}\\
&= \frac{\exp(\beta V_{t+2}^i - \beta V_{t+1}^i)^{1/\nu}}{D}\\
&= \frac{(\hat{u}_{t+2}^i)^{\beta/\nu}}{D}
\end{aligned}$$

where the denominator of the expression is

$$D = \frac{\sum_{j=1}^N\ \exp(\beta V_{t+2}^j\ -\ \tau^{n,j})^{1/\nu}}{\sum_{k=1}^N\ \exp(\beta V_{t+1}^k\ -\ \tau^{n,k})^{1/\nu}}$$

Expanding each term in the numerator by $\exp(\beta V_{t+1}^{j} \;-\; \tau^{n,j})^{1/\nu}$, we have

$$
\begin{aligned}
D \;&=\; \frac{\sum_{j=1}^{N} \; \exp(\beta V_{t+2}^{j} \;-\; \tau^{n,j})^{1/\nu} \, \frac{\exp(\beta V_{t+1}^{j} - \tau^{n,j})^{1/\nu}}{\exp(\beta V_{t+1}^{j} - \tau^{n,j})^{1/\nu}}}{\sum_{k=1}^{N} \; \exp(\beta V_{t+1}^{k} \;-\; \tau^{n,k})^{1/\nu}} \\
&=\; \sum_{j=1}^{N} \frac{\exp(\beta V_{t+2}^{j} \;-\; \tau^{n,j})^{1/\nu}}{\exp(\beta V_{t+1}^{j} \;-\; \tau^{n,j})^{1/\nu}} \frac{\exp(\beta V_{t+1}^{j} \;-\; \tau^{n,j})^{1/\nu}}{\sum_{k=1}^{N} \; \exp(\beta V_{t+1}^{k} \;-\; \tau^{n,k})^{1/\nu}} \\
&=\; \sum_{j=1}^{N} (\hat{u}_{t+2}^{j})^{\beta/\nu} \; \mu_t^{n,j}
\end{aligned}
$$

and hence

$$
\frac{\mu_{t+1}^{n,i}}{\mu_t^{n,i}} \;=\; \frac{(\hat{u}_{t+2}^{i})^{\beta/\nu}}{\sum_{j=1}^{N} (\hat{u}_{t+2}^{j})^{\beta/\nu} \; \mu_t^{n,j}}
$$

which gives the result.

- To derive the equation (2), note that

$$
\begin{aligned}
V_{t+1}^{n} \;-\; V_t^{n} \;&=\; \ln \frac{w_{t+1}^{n}}{P_{t+1}^{n}} \;-\; \ln \frac{w_t^{n}}{P_t^{n}} \\
&\quad +\; \nu \; \ln \left(\sum_{i=1}^{N} \exp(\beta V_{t+2}^{i} \;-\; \tau^{n,i})^{1/\nu} \right) \;-\; \nu \; \ln \left(\sum_{j=1}^{N} \exp(\beta V_{t+1}^{j} \;-\; \tau^{n,j})^{1/\nu} \right) \\
&=\; \ln \frac{w_{t+1}^{n}}{P_{t+1}^{n}} \;-\; \ln \frac{w_t^{n}}{P_t^{n}} \;+\; \nu \ln \frac{\sum_{i=1}^{N} \exp(\beta V_{t+2}^{i} \;-\; \tau^{n,i})^{1/\nu}}{\sum_{j=1}^{N} \exp(\beta V_{t+1}^{j} \;-\; \tau^{n,j})^{1/\nu}}
\end{aligned}
$$

Expanding each term in the numerator by $\exp(\beta V_{t+1}^{i} \;-\; \tau^{n,i})^{1/\nu}$, we have

$$
V_{t+1}^{n} \;-\; V_t^{n} \;=\; \ln \frac{w_{t+1}^{n}}{P_{t+1}^{n}} \;-\; \ln \frac{w_t^{n}}{P_t^{n}} \;+\; \nu \; \ln \left(\sum_{i=1}^{N} \exp \mu_t^{n,i} \; (\beta V_{t+2}^{i} \;-\; \beta V_{t+1}^{i})^{1/\nu} \right)
$$

Taking the exponential on both sides of the equation, yields:

$$
\hat{u}_{t+1}^{n} \;=\; \frac{\hat{w}_{t+1}^{n}}{\hat{P}_{t+1}^{n}} \left(\sum_{i=1}^{N} \mu_t^{n,i} \; (\hat{u}_{t+2}^{i})^{\beta/\nu} \right)^{\nu}
$$

which proves the result.

3 Closing the Model

3.1 The EK Model without Intermediate Goods

- To close the model would be to specify a static trade model that gives us a system of equations to solve for wages w_t^n and price levels P_t^n as a function of fundamentals such as iceberg costs and location specific productivities. We can solve a sequence of the equilibria using static hat algebra as long as we know the law of motion for the populations.

- Consider the EK model without intermediate goods as discussed in the MIT lecture notes. We only need to add time subscripts t to the notation.

- Recall, that each region i at time t has a distribution of productivities given by:

$$F_t^i(z) = \exp(-T_t^i z^{-\theta})$$

- $d_t^{ni} \geq 1$ units need to be shipped from i so that 1 unit makes it to n (iceberg costs).

- In the model without intermediate goods $c_t^i = w_t^i / P_t^i$ because of constant returns to scale.

- Recall that the price distribution faced by region n is

$$G_t^n(p) = 1 - \exp(-\Phi_t^n p^\theta)$$

  where

$$\Phi_t^n = \sum_{i=1}^{N} T_t^i \left(w_t^i d_t^{ni}\right)^{-\theta}$$

  which is also the price distribution of prices that region n buys from any region i, i.e. all exporters face the same price distribution in equilibrium.

- Region n buys good from region i if $i = \arg\min\{p_1, ..., p_N\}$ and the shares are given by

$$\pi_t^{ni} = \frac{T_t^i (w_t^i d_t^{ni})^{-\theta}}{\Phi_t^n}$$

- The overall price index in region n is

$$\begin{aligned} P_t^n &= \left(\int_0^1 p(u)^{1-\sigma} \, dG_t^n(u) \right)^{\frac{1}{1-\sigma}} \\ &= \gamma \, (\Phi_t^n)^{-\frac{1}{\theta}} \\ &= \gamma \, \left(\sum_{i=1}^N T_t^i \, (w_t^i d_t^{ni})^{-\theta} \right)^{-\frac{1}{\theta}} \end{aligned}$$

where

$$\gamma = \left[\Gamma \left(\frac{1-\sigma}{\theta} + 1 \right) \right]^{1/(1-\sigma)}$$

- In this simple model without intermediate goods the price index in region n is a function of the exogenous fundamentals as well as the endogenous wages in all regions.

- Let X_t^{ni} be total spending of region n on goods from region i.

- Total spending of region n is $X_t^n = \sum_{i=1}^N X_t^{ni}$.

- Then

$$\frac{X_t^{ni}}{X_t^n} = \pi_t^{ni} = \frac{T_t^i (w_t^i d_t^{ni})^{-\theta}}{\Phi_t^n}$$

or

$$X_t^{ni} = \frac{T_t^i (w_t^i d_t^{ni})^{-\theta}}{\Phi_t^n} \, X_t^n$$

- In equilibrium, income of region n must equal total expenditures on goods from region n (zero trade deficit):

$$Y_t^n \;=\; w_t^n L_t^n = \sum_{i=1}^{N} X_t^{in}$$

and hence we have:

$$\begin{aligned} w_t^n L_t^n &= \sum_{i=1}^{N} X_t^{in} \\ &= \sum_{i=1}^{N} \frac{T_t^n (w_t^n d_t^{in})^{-\theta}}{\sum_{j=1}^{N} T_t^j \; (w_t^j d_t^{ij})^{-\theta}} \; w_t^i L_t^i \end{aligned}$$

These are $N-1$ equations in N unknowns. Hence we can compute the equilibrium wages from that system of equations up to a normalization if we know $L_t^1, ...L_t^N$.

3.2 Comparative Statics and Hat Algebra

- Finally, recall from the previous lecture on the EK model that we can solve for a sequence of static equilibria using static hat algebra.

- In particular, we have:

$$\hat{w}_{t+1}^n \hat{L}_{t+1}^n Y_t^n \;=\; \sum_{i=1}^{N} \frac{\pi_t^{in} \; \hat{T}_{t+1}^n \; (\hat{w}_{t+1}^n \hat{d}_{t+1}^{in})^{-\theta}}{\sum_{j=1}^{N} \pi_t^{ij} \; \hat{T}_{t+1}^j \; (\hat{w}_{t+1}^j \hat{d}_{t+1}^{ij})^{-\theta}} \; \hat{w}_{t+1}^i \hat{L}_{t+1}^i \; Y_t^i \tag{4}$$

and

$$\hat{P}_{t+1}^n \;=\; \left(\sum_{i=1}^{N} \pi_t^{ni} \; \hat{T}_{t+1}^i \; (\hat{w}_{t+1}^i \hat{d}_t^{ni})^{-\theta} \right)^{-\frac{1}{\theta}} \tag{5}$$

and

$$\hat{\pi}_{t+1}^{ni} \;=\; \frac{\hat{T}_{t+1}^i (\hat{w}_{t+1}^i \hat{d}_{t+1}^{ni})^{-\theta}}{\sum_{j=1}^{N} \pi_t^{nj} \hat{T}_{t+1}^j \; (\hat{w}_{t+1}^j \hat{d}_{t+1}^{nj})^{-\theta}} \tag{6}$$

3.3 Autarkie

- In the Autarkie equilibrium, trade-costs are prohibitively high.
- Since there is no trade, the price index is given by:

$$
\begin{aligned}
P_t^n &= \left(\int_0^1 p(u)^{1-\sigma} du\right)^{\frac{1}{1-\sigma}} \\
&= \gamma \left(T_t^n \left(w_t^n\right)^{-\theta}\right)^{-\frac{1}{\theta}}
\end{aligned}
$$

- If total spending on manufacturing goods is exogenously given and labor supply in manufacturing is fixed, then wages satisfy:

$$
w_t^n = \frac{X_t^n}{L_t^n}
$$

4 The Main Result of CDP

- Given an initial allocation of labor $L_0^1, ...L_0^N$ and wages $w_0^1, ...w_0^N$ and an exogenous sequence of changes in fundamentals (iceberg costs, $\hat{d}_1, \hat{d}_2, ...\hat{d}_{\bar{t}}$, and productivities, $\hat{T}_1, \hat{T}_2, ...\hat{T}_{\bar{t}}$), assume that the economy reaches a steady state in $\bar{t}$ and $\hat{u}_t^i = 1$ for all $t \geq \bar{t}$. Then the evolution of the economy is completely characterized by a solution to the system of equations given by equations (1) - (6).
- This result then implies that we can compute an equilibrium using the following algorithm.
- Take as given the set of initial conditions $(L_0, \mu_{-1}, w_0, P_0, \pi_0)$. Note that given L_0 and w_0, we know Y_0.

  Suppose we are at iteration step k, update all variables as follows:

1. For all $t \geq 0$, use $\left\{\hat{u}^n_{t+1}(k)\right\}_{t=0}^{\bar{t}}$ and $\mu^{n,i}_{-1}$ to solve for the path of $\left\{\mu^{n,i}_t(k+1)\right\}_{t=0}^{T}$ using equation (1).
2. Use the path for $\left\{\mu^{n,i}_t(k+1)\right\}_{t=0}^{T}$ and L^n_0 to get the path for $\left\{L^n_{t+1}(k+1)\right\}_{t=0}^{T}$ using equation (3).
3. Given the sequence $\left\{L^n_{t+1}(k+1)\right\}_{t=0}^{\bar{t}}$, Y_0, π_0, use equations (4) - (6) to compute the sequences of $\left\{\hat{w}^n_{t+1}(k+1)\right\}_{t=0}^{\bar{t}}$, $\left\{\hat{P}^n_{t+1}(k+1)\right\}_{t=0}^{\bar{t}}$ and $\left\{\hat{\pi}^n_{t+1}(k+1)\right\}_{t=0}^{\bar{t}}$.
4. For each t, use $\mu^{n,i}_t$(k+1), $\hat{w}^n_{t+1}(k+1)$, $\hat{P}^n_{t+1}(k+1)$, and $\hat{u}^n_{t+2}(k)$ to solve backwards for $\hat{u}^n_{t+1}(k+1)$ using equation (2). This delivers a new path for $\left\{\hat{u}^n_{t+1}(k+1)\right\}_{t=0}^{\bar{t}}$.
5. Check if $\left\{\hat{u}^n_{t+1}(k+1)\right\}_{t=0}^{\bar{t}} \simeq \left\{\hat{u}^n_{t+1}(k)\right\}_{t=0}^{\bar{t}}$. The path should converge to $\hat{u}^n_{\bar{t}+1} = 1$ for a sufficiently large $\bar{t}$. If it is not, continue to iterate